Lecture 23

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1 Properties of determinants-2

Now we'll give a first motivation of the determinant.

Theorem 1.1 (Criteria of invertibility). *A square matrix A is invertible if and only if* $\det A \neq 0$.

Proof. Let's use elementary row operations to transform a matrix A to its triangular (row-echelon) form. Let's note, that if the determinant was not equal to 0, then it will not be equal to 0 after elementary row operations, and if it was equal to 0, it will be equal to 0 after elementary row operations. The matrix A is invertible, if it's REF doesn't have a row of zeros, i.e. the determinant of its REF is not equal to 0. So, the determinant of the initial matrix is not equal to 0. Moreover, A is not invertible if we have a row of zeros in its REF, so the determinant of REF equals to 0, and so, the determinant of the initial matrix A is equal to 0. □

Example 1.2. *We computed the following determinant:*

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$$

So, this matrix is not invertible.

This theorem gives us a criteria, when the matrix is invertible. But if we know that the determinant is not equal to 0, and so the matrix is invertible, the theorem doesn't give us a method of computing the inverse.

Now let's continue with properties of determinants.

Theorem 1.3 (The determinant of the transpose). $\det A^\top = \det A$.

Example 1.4. *Since* $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$, *we have that* $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0$.

Proof. The determinant of $A^\top$ is equal to the sum of all possible products of matrix elements, taken 1 from each column and 1 from each row, as well as the determinant of A. So we have to check that the products are included with the same signs.

To figure out the sign before $a_{1k_1}a_{2k_2}\dots a_{3k_3}$ in the expression for the determinant of $A^\top$, we have to reorder multiplicands by the second subscript. So, we can interchange multiplicands, and transpositions will occur simultaneously in the first subscripts and in the second subscripts. So, the sign of the final permutation will be the same, i.e. if at the end we'll get $a_{l_1 1}a_{l_2 2}\dots a_{l_n n}$, then

$$\operatorname{sgn}(k_1, k_2, \dots, k_n) = \operatorname{sgn}(l_1, l_2, \dots, l_n),$$

and so, $\det A = \det A^\top$. □

From this theorem it follows that all properties which hold for rows of the matrix, hold for its columns, i.e.

- Interchanging of columns changes the sign of the determinant;
- If two columns are equal, the determinant is equal to 0;
- If there is a column of 0's, then the determinant is equal to 0.
- If we multiply a column by c, the determinant is multiplied by c as well;
- If we add r's column multiplied by c to the s's column, the determinant will not change.

Theorem 1.5 (Determinant of the product). *For any square matrices A and B*

$$\det(AB) = \det A \det B.$$

Remark 1.6. *For the addition this fact is not true:*

$$\det(A + B) \neq \det A + \det B.$$

The proof of this theorem is quite complicated, and you will be able to find it in the addendum to this lecture. It is too theoretical, and requires additional general theorem which we will not state here, but it will be stated and proved in addendum.

Example 1.7. *Let* $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$, *then* $\det A = 2 \cdot 5 - 1 \cdot 3 = 7$. *Let* $B = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$, *so* $\det B = 1 \cdot (-4) - (-2) \cdot 3 = 2$. *Now*

$$AB = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 18 & -26 \end{pmatrix}$$

and so $\det AB = 5 \cdot (-26) - (-8) \cdot 18 = -130 + 144 = 14 = 7 \cdot 2 = \det A \det B$.

Theorem 1.8 (The determinant of a block matrix). *Let A be a matrix of the following form:*

$$A = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix}$$

where B and C are square matrices. Then

$$\det A = \det B \det C.$$

We will not proof this theorem here either, since the proof requires the same general fact as the previous theorem. The proof can be found in the addendum.

Example 1.9. *Let*

$$A = \begin{pmatrix} 1 & 2 & 76 & 128 \\ 3 & 8 & -512 & 872 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -5 & -3 \end{pmatrix}$$

Here

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}, \text{ and } C = \begin{pmatrix} -2 & 3 \\ -5 & -3 \end{pmatrix}$$

so $\det B = 1 \cdot 8 - 2 \cdot 3 = 2$, *and* $\det C = (-2) \cdot (-3) - 3 \cdot (-5) = 21$, *so* $\det A = 2 \cdot 21 = 42$.

Corollary 1.10. *Let A be a matrix of the following form:*

$$A = \begin{pmatrix} B & 0 \\ D & C \end{pmatrix}$$

where B and C are square matrices. Then

$$\det A = \det B \det C.$$

The following example will be very useful in the future — it will be used extensively in the next algorithm of computing the determinant.

Example 1.11. *Let*

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

Then we can consider this matrix as a block matrix $A = \begin{pmatrix} B & 0 \\ D & C \end{pmatrix}$ with the following blocks:

$$B = \begin{pmatrix} a_{11} \end{pmatrix}; \quad D = \begin{pmatrix} a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{pmatrix}; \quad C = \begin{pmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

So,

$$\det A = a_{11} \det C.$$

Now we will give 2 definitions. Let A be an $n \times n$-matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Definition 1.12. *The **minor** M_{ij} of the element a_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and j-th column of a matrix A. The **cofactor** A_{ij} of the element a_{ij} is $(-1)^{i+j}M_{ij}$.*

So we see that cofactors are equal to minors by absolute value, but the sign is different. The signs are distributed by checkerboard pattern:

$$\begin{pmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example 1.13. *Let* $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. *Then*

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3; \quad M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6; \quad M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3;$$

$$M_{21} = \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -6; \quad M_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = -12; \quad M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6;$$

$$M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3; \quad M_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -6; \quad M_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3.$$

Now,

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3; \quad A_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 6; \quad A_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3;$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = 6; \quad A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = -12; \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 6;$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3; \quad A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6; \quad A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3.$$

Lemma 1.14.

$$\begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_{ij} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix} = a_{ij}A_{ij}$$

Example 1.15.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 8 & 9 \end{vmatrix} = 5A_{22} = 5(-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 5(1 \cdot 9 - 3 \cdot 7) = 5(-12) = -60.$$

Proof. Let's interchange i-th row with all previous rows, and j-th column with all previous columns. So, there will be $i-1$ row interchanges, and $j-1$ column interchanges, so the determinant will be multiplied by $(-1)^{i-1+j-1} = (-1)^{i+j}$. At the end of this process we'll get the following determinant:

$$\begin{vmatrix} a_{ij} & 0 & \dots & 0 \\ a_{1j} & a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{nj} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

By the example 1.11 it is equal to $a_{ij}M_{ij}$, so the determinant of the initial matrix is equal to $(-1)^{i+j}a_{ij}M_{ij} = a_{ij}A_{ij}$. □

The following theorem gives the algorithm of computing the determinants.

Theorem 1.16. *[Expansion by a row(column)] For any square matrix A*

$$\det A = \sum_j a_{ij}A_{ij} = \sum_i a_{ij}A_{ij}, \quad i.e$$

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$$

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}$$

Example 1.17. *Let's compute the determinant* $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$. *We will use different expansions and, of course, we will get the same result.*

Expansion by the 1st row

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$\begin{aligned} &= 1(45-48) - 2(36-42) + 3(32-35) \\ &= (-3) - 2(-6) + 3(-3) \\ &= -3 + 12 - 9 \\ &= 0. \end{aligned}$$

Expansion by the 2nd row

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$\begin{aligned} &= -4(18-24) + 5(9-21) - 6(8-14) \\ &= (-4)(-6) + 5(-12) - 6(-6) \\ &= 24 - 60 + 36 \\ &= 0. \end{aligned}$$

Expansion by the 1st column

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$\begin{aligned} &= (42-48) - 4(18-24) + 7(12-15) \\ &= (-3) - 4(-6) + 7(-3) \\ &= -3 + 24 - 21 \\ &= 0. \end{aligned}$$

Expansion by the 2nd column

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$\begin{aligned} &= (-2)(36-42) + 5(9-21) - 8(6-12) \\ &= (-2)(-6) + 5(-12) - 8(-6) \\ &= 12 - 60 + 48 \\ &= 0. \end{aligned}$$

Proof of the theorem 1.16 (Row (column) expansion). We'll prove this formula for row expansion, the proof of the column expansion is similar. The i-th row of a matrix A is equal to the sum of rows $(a_{i1}, 0, \ldots, 0) + (0, a_{i2}, \ldots, 0) + \cdots + (0, 0, \ldots, a_{in})$. Using the property of multilinearity, we have that

$$\begin{aligned}
\det A &= \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \\
&= \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{i1} & 0 & \ldots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ 0 & a_{i2} & \ldots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \ldots & a_{in} \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \\
&= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\
&= \sum_j a_{ij}A_{ij}.
\end{aligned}$$

$\square$